Quiz 1: MATH 212 (Introductory PDEs)

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Duration: 70 minutes

Name (Last, First):

Student number: _

Section 1: 11am-12pm (Sophie Moufawad)

Section 2: 12pm-01pm (Wael Mahboub)

Section 3: 01pm-02pm (Wael Mahboub)

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Problem	Score
1	8 /10
2	18 /18
3	15 /15
4	20 /20
5	35 /37
Total	96/100

[10 points=7+3] **Problem 1.** Consider the following boundary value problem

$$u_t + 3u_x - u = 0, \quad t > 0, \; 0 \le x \le 3 \qquad (1)$$
$$u(t,0) = e^{2t}, \qquad (2)$$
$$u(t,3) = e^{2t-1}. \qquad (3)$$

(a) Find all the separable eigensolutions of the damped uniform transport equation (1).

8

Let $u(t,x) = w(t)\,v(x)$

$u_t = w'v$ & $u_x = wv'$

subs in the (1): $w'v + 3wv' - wv = 0$

$w'v = -3wv' + wv$

$w'v = w(-3v' + v)$

$\Rightarrow \frac{w'}{w} = \frac{-3v'+v}{v} = \lambda$

So, $\begin{cases} w' - \lambda w = 0 \;\cdots\; (4) \\ -3v' + v = \lambda v \Rightarrow -3v' + (1-\lambda)v = 0 \;\cdots\; (5) \end{cases}$

(4) yields $w(t) = e^{\lambda t}$.

(5) yields $v' + \frac{\lambda'-1}{3} v = 0$

$\Rightarrow v(x) = e^{-\frac{\lambda'-1}{3}x}$

the same λ!

So, $u(t,x) = e^{\lambda t} e^{-\frac{\lambda'-1}{3}x}$

$= e^{\lambda t - \frac{\lambda'-1}{3}x}$ $\quad \begin{pmatrix} \lambda \in \mathbb{R} \\ \lambda' \in \mathbb{R} \end{pmatrix}$

(b) Find the separable eigensolutions that satisfy the boundary conditions (2), and (3).

$u(t,0) = e^{\lambda t} = e^{2t}$

& $u(t,3) = e^{\lambda t - \frac{\lambda'-1}{3}x} = e^{\lambda t - \lambda' + 1} = e^{2t-1}$

$\Rightarrow \begin{cases} \lambda t = 2t \Rightarrow \lambda = 2 \\ \cancel{\lambda t} - \lambda' + 1 = \cancel{2t} - 1 \Rightarrow \lambda' = 2 \end{cases}$

So, $u(t,x) = e^{2t - \frac{x}{3}}$

[18 points] **Problem 2.** Solve the following initial value problem

$$\begin{cases} u_x + \frac{4x^2}{t} u_t = 0, \quad t > 0, \ x > 0 \\ u(0, x) = \frac{1}{1+x^3}. \end{cases}$$

$$\cancel{\frac{dx}{dt} = \frac{4x^2}{t}}$$

$$\cancel{\int \frac{dx}{4x^2} = \int \frac{dt}{t}}$$

$$\cancel{-\frac{1}{4x} = \ln|t| + c}$$

$$\cancel{\Rightarrow c = -\frac{1}{4x} - \ln|t| = -\frac{1}{4x} - \ln t \ (\text{since } t>0)}$$

$$\cancel{\Rightarrow u(t,x) = f\left(-\frac{1}{4x} - \ln|t|\right)}$$

$$\cancel{u(0,x) = f}$$

the PDE can be written as $u_t + \frac{t}{4x^2} u_x = 0$

$$\frac{dx}{dt} = \frac{t}{4x^2}$$

$$\int 4x^2 \, dx = \int t \, dt$$

$$\frac{4}{3} x^3 = \frac{t^2}{2} + c$$

$$c = \frac{4}{3} x^3 - \frac{t^2}{2} \quad \text{(characteristic curves)}$$

$$\Rightarrow u(t,x) = f\left(\frac{4}{3} x^3 - \frac{t^2}{2}\right) \quad \text{(u is constant along the curves)}$$

$$u(0,x) = f\left(\frac{4}{3} x^3\right) = \frac{1}{1+x^3}$$

$$f(x) = \frac{1}{1 + \frac{3}{4} x}$$

So, $u(t,x) = \dfrac{1}{1 + \frac{3}{4}\left(\frac{4}{3} x^3 - \frac{t^2}{2}\right)} = \dfrac{1}{1 + x^3 - \frac{3}{8} t^2}$

[15 points] **Problem 3.** Solve the following initial value problem

$$\begin{cases} u_{tt} = 9u_{xx}, \ t > 0 & \cdots \ c = 3 \\ u(0,x) = \begin{cases} 2 & if \ 3 < x < 4 \\ 0, & otherwise \end{cases} \\ u_t(0,x) = 0. \end{cases}$$

d'Alembert's: $u(t,x) = \dfrac{f(x+3t)+f(x-3t)}{2} + \dfrac{1}{2(3)} \displaystyle\int_{x-3t}^{x+3t} g(z)\,dz$

where $f(x) = u(0,x)$ & $g(x) = u_t(0,x)$

in this case, $g(x) = 0 \Rightarrow u(t,x) = \dfrac{f(x+3t)+f(x-3t)}{2}$

$f(x+3t) = \begin{cases} 2, & 3 < x+3t < 4 \\ 0, & \text{otherwise} \end{cases}$ & $f(x-3t) = \begin{cases} 2, & 3 < x-3t < 4 \\ 0, & \text{otherwise} \end{cases}$

$\Rightarrow f(x+3t) = \begin{cases} 2, & 3-3t < x < 4-3t \\ 0, & \text{otherwise} \end{cases}$ & $f(x-3t) = \begin{cases} 2, & 3+3t < x < 4+3t \\ 0, & \text{otherwise} \end{cases}$

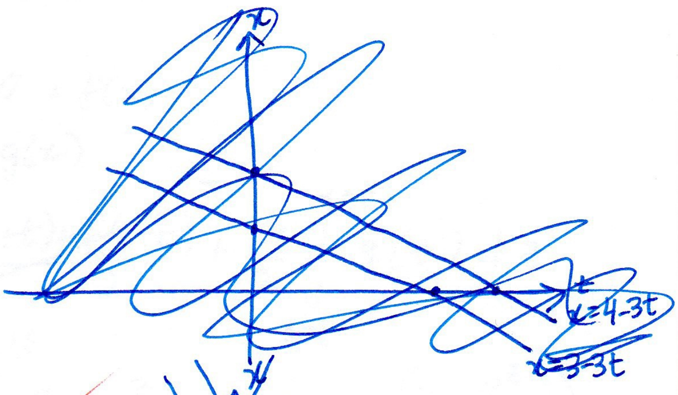
in ①: $f(x+3t) = f(x-3t) = 0$

in ②: $f(x+3t) = 2$ & $f(x-3t) = 0$

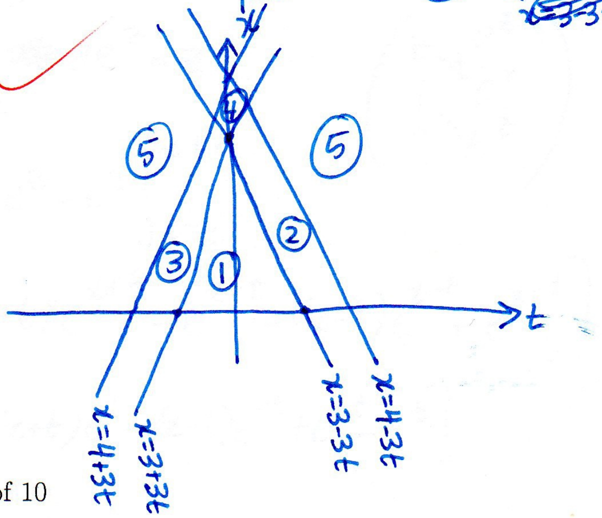
in ③: $f(x+3t) = 0$ & $f(x-3t) = 2$

in ④: $f(x+3t) = f(x-3t) = 2$

in ⑤: $f(x+3t) = f(x-3t) = 0$



So, $u(t,x) = \begin{cases} 0, & \text{for } ① \ \& \ ⑤ \\ 1, & \text{for } ② \ \& \ ③ \\ 2, & \text{for } ④ \end{cases}$



[20 points=10+7+3] Problem 4. The goal of this problem is to solve the following initial value problem.

$$\begin{cases} u_{tt} = u + 2u_x + u_{xx} & t > 0 \\ u(0,x) = 0, \forall x \in \mathbb{R} \\ u_t(0,x) = x \end{cases}$$

(a) Let $v(t,x) = e^x u(t,x)$. Show that $v(t,x)$ satisfies a wave equation, to be determined.

$u = e^{-x} v$

$u_x = -e^{-x} v + e^{-x} v_x$

$u_{xx} = e^{-x} v - e^{-x} v_x - e^{-x} v_x + e^{-x} v_{xx} = e^{-x} v - 2e^{-x} v_x + e^{-x} v_{xx}$

$u_t = e^{-x} v_t$

$u_{tt} = e^{-x} v_{tt}$

subs in the PDE: $e^{-x} v_{tt} = \cancel{e^{-x} v} \cancel{-2e^{-x} v} \cancel{+2e^{-x} v_x} \cancel{+e^{-x} v} \cancel{-2e^{-x} v_x} + e^{-x} v_{xx}$

$e^{-x} v_{tt} = e^{-x} v_{xx}$

$\Rightarrow v_{tt} = v_{xx}$

So, $v(t,x)$ satisfies $v_{tt} = v_{xx}$. $(c=1)$

(b) Find $v(t,x)$ that solves the wave equation found in (a), along with suitable initial conditions.

$v(0,x) = e^x u(0,x) = e^x (0) = 0 = f(x)$

$v_t(0,x) = e^x u_t(0,x) = xe^x = g(x)$

D'Alembert's: $v(t,x) = \cancel{\dfrac{f(x+t)+f(x-t)}{2}} + \dfrac{1}{2}\displaystyle\int_{x-t}^{x+t} g(z)\,dz$ 0

$= \dfrac{1}{2}\displaystyle\int_{x-t}^{x+t} ze^z\,dz$

$$\begin{array}{ll} z & e^z \\ 1 & e^z \\ 0 & e^z \end{array}$$

$= \dfrac{1}{2}\left[ze^z - e^z\right]_{x-t}^{x+t}$

$= \dfrac{1}{2}\left[(x+t)e^{x+t} - e^{x+t} - (x-t)e^{x-t} + e^{x-t}\right]$

$= \dfrac{e^x}{2}\left[(x+t)e^t - (x-t)e^{-t} + e^{-t} - e^t\right]$

(c) Deduce $u(t, x)$.

$$u(t,x) = e^{-x} v(t,x)$$

$$= e^{-x} \cdot \frac{e^{x}}{2}\left[(x+t)e^{t} - (x-t)e^{-t} + e^{-t} - e^{t}\right]$$

$$= \frac{1}{2}\left((x+t)e^{t} - (x-t)e^{-t} + e^{-t} - e^{t}\right)$$

[37 points=5+7+10+15] **Problem 5.** The goal of this problem is to solve the following initial value problem

$$\begin{cases} u_{tt} - u_{tx} - 6u_{xx} = sin(x+t), & t > 0, \ \forall x \in \mathbb{R} \\ u(0,x) = x^2, \\ u_t(0,x) = e^{2x} \end{cases} \tag{4}$$

(a) Find a particular solution to the inhomogeneous pde $u_{tt} - u_{tx} - 6u_{xx} = sin(x+t)$ of the form $u_p(t,x) = c_1 sin(x+t) + c_2 cos(x+t)$, where $c_1, c_2 \in \mathbb{R}$.

$u_p = c_1\sin(x+t) + c_2\cos(x+t)$

$u_{p_t} = c_1\cos(x+t) - c_2\sin(x+t)$

$u_{p_{tt}} = -c_1\sin(x+t) - c_2\cos(x+t)$

$u_{p_{tx}} = -c_1\sin(x+t) - c_2\cos(x+t)$

$u_{p_x} = c_1\cos(x+t) - c_2\sin(x+t)$

$u_{p_{xx}} = -c_1\sin(x+t) - c_2\cos(x+t)$

Subs in the pde:

$\cancel{-c_1\sin(x+t)} \cancel{- c_2\cos(x+t)} + \cancel{c_1\sin(x+t)} + \cancel{c_2\cos(x+t)}$

$+6c_1\sin(x+t) + 6c_2\cos(x+t) = \sin(x+t)$

$\Longrightarrow \begin{cases} 6c_1 = 1 \\ 6c_2 = 0 \end{cases} \Longrightarrow \begin{cases} c_1 = \frac{1}{6} \\ c_2 = 0 \end{cases}$

✓

So, $u_p(t,x) = \frac{1}{6}\sin(x+t)$

(b) Find a solution to the homogeneous pde $u_{tt} - u_{tx} - 6u_{xx} = 0$ by factorizing the corresponding differential operator.

$\boxed{\text{(b) in page 9.}}$

~~the pde is similar to the equation~~ $\cancel{a^2 - ab - 6b^2 = 0}$

$\cancel{(a+2b)(a-3b) = 0}$

$\cancel{\Rightarrow \left(\frac{\partial}{\partial t} + 2\frac{\partial}{\partial x}\right)\left(\frac{\partial u}{\partial t} - 3\frac{\partial u}{\partial x}\right) = 0}$ ~~$\Rightarrow$ So, the solution to the homogeneous pde is~~ $\cancel{u(t,x) = f(x-2t) + g(x+3t)}$.

(c) Let $\varphi = x - 2t$ & $\eta = x + 3t$

$v(\varphi, \eta) = u(t, x)$

So, $u_t = \frac{\partial u}{\partial t} = \frac{\partial v}{\partial \varphi}\frac{\partial \varphi}{\partial t} + \frac{\partial v}{\partial \eta}\frac{\partial \eta}{\partial t} = -2v_\varphi + 3v_\eta$

$u_{tt} = -2\frac{\partial v_\varphi}{\partial \varphi}\frac{\partial \varphi}{\partial t} - 2\frac{\partial v_\varphi}{\partial \eta}\frac{\partial \eta}{\partial t} + 3\frac{\partial v_\eta}{\partial \eta}\frac{\partial \eta}{\partial t} + 3\frac{\partial v_\eta}{\partial \varphi}\frac{\partial \varphi}{\partial t}$

$= 4v_{\varphi\varphi} - 6v_{\varphi\eta} + 9v_{\eta\eta} - 6v_{\eta\varphi}$ $\quad (v_{\varphi\eta} = v_{\eta\varphi})$

$= 4v_{\varphi\varphi} + 9v_{\eta\eta} - 12v_{\varphi\eta}$

$u_x = \frac{\partial u}{\partial x} = \frac{\partial v}{\partial \varphi}\frac{\partial \varphi}{\partial x} + \frac{\partial v}{\partial \eta}\frac{\partial \eta}{\partial x} = v_\varphi + v_\eta$

$u_{xx} = \frac{\partial v_\varphi}{\partial \varphi}\frac{\partial \varphi}{\partial x} + \frac{\partial v_\varphi}{\partial \eta}\frac{\partial \eta}{\partial x} + \frac{\partial v_\eta}{\partial \varphi}\frac{\partial \varphi}{\partial x} + \frac{\partial v_\eta}{\partial \eta}\frac{\partial \eta}{\partial x}$

$= v_{\varphi\varphi} + 2v_{\varphi\eta} + v_{\eta\eta}$

$u_{tx} = -2\frac{\partial v_\varphi}{\partial \varphi}\frac{\partial \varphi}{\partial x} - 2\frac{\partial v_\varphi}{\partial \eta}\frac{\partial \eta}{\partial x} + 3\frac{\partial v_\eta}{\partial \varphi}\frac{\partial \varphi}{\partial x} + 3\frac{\partial v_\eta}{\partial \eta}\frac{\partial \eta}{\partial x}$

$= -2v_{\varphi\varphi} + v_{\varphi\eta} + 3v_{\eta\eta}$

subs in the pde:

$\cancel{4v_{\varphi\varphi}} + \cancel{9v_{\eta\eta}} - 12v_{\varphi\eta} + \cancel{2v_{\varphi\varphi}} - v_{\varphi\eta} - \cancel{3v_{\eta\eta}} - \cancel{6v_{\varphi\varphi}} - 12v_{\varphi\eta} - \cancel{6v_{\eta\eta}} = 0$

$-25v_{\varphi\eta} = 0$

$\Rightarrow v_{\varphi\eta} = 0$

$$\frac{\partial}{\partial \eta}\left(\frac{\partial v}{\partial \varphi}\right) = 0$$

$\frac{\partial v}{\partial \varphi} = B(\eta)$ which is some function of η.

$\Rightarrow v(\varphi, \eta) = f(\varphi) + g(\eta)$

where $f(\varphi)$ & $g(\eta)$ are random functions

$(g(\eta) = B^{*}(\eta))$

So, $u(t, x) = f(x-2t) + g(x+3t)$

(c) Show that all the solutions to the homogeneous pde $u_{tt} - u_{tx} - 6u_{xx} = 0$ have the same form as the solution found in part (b). (Hint: apply the change of variable from (t, x) to (μ, ξ), where $u(t, x) = v(\mu, \xi)$.)

part (c) is done previously in part (b)

(with μ denoted as φ
& φ denoted as η)

(b) the pde is similar to $a^2 - ab - 6b^2 = 0$
$(a+2b)(a-3b) = 0$

$$\left(\frac{\partial}{\partial t} + 2\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - 3\frac{\partial}{\partial x}\right)[u] = 0$$

$$L_1 = \frac{\partial}{\partial t} + 2\frac{\partial}{\partial x} \quad \& \quad L_2 = \frac{\partial}{\partial t} - 3\frac{\partial}{\partial x}$$

Both operators give each solution to the pde.

$\Rightarrow$ the solution to this pde is

$$u(t,x) = f(x-2t) + g(x+3t)$$

which is the summation of both solutions from both pdes $L_1[u] = 0$ & $L_2[u] = 0$.

(We took the summation due to linearity!)

(d) Solve the initial value problem (4).

$$u(t,x) = f(x-2t) + g(x+3t)$$

$$u(0,x) = f(x) + g(x) = x^2 \quad \cdots \text{①}$$

$$u_t(t,x) = -2f'(x-2t) + 3g'(x+3t)$$

$$u_t(0,x) = -2f'(x) + 3g'(x) = e^{2x} \quad \cdots \text{②}$$

deriving ① wrt x: $f'(x) + g'(x) = 2x \quad \cdots$ ③

② + 2x③ yields $3g'(x) + 2g'(x) = e^{2x} + 4x$

$$5g'(x) = e^{2x} + 4x$$

$$g'(x) = \frac{e^{2x}}{5} + \frac{4}{5}x$$

$$\Rightarrow g(x) = \frac{2e^{2x}}{5} + \frac{2}{5}x^2 + K \quad (K \in \mathbb{R})$$

from ①: $f(x) = x^2 - g(x)$

$$= x^2 - \frac{2}{5}e^{2x} - \frac{2}{5}x^2 - K$$

$$= -\frac{2}{5}e^{2x} + \frac{3}{5}x^2 - K$$

So, $u(t,x) = -\frac{2}{5}e^{2(x-2t)} + \frac{3}{5}(x-2t)^2 \cancel{-K} + \frac{2}{5}e^{2(x+3t)} + \frac{2}{5}(x+3t)^2 \cancel{+K}$

$$= -\frac{2}{5}e^{2(x-2t)} + \frac{2}{5}e^{2(x+3t)} + \frac{3}{5}(x-2t)^2 + \frac{2}{5}(x+3t)^2$$

This is the solution of the homogeneous PDE.

~~The general solution is:~~

The solution of the IVP:

$$u(t,x) = -\frac{2}{5}e^{2(x-2t)} + \frac{2}{5}e^{2(x+3t)} + \frac{3}{5}(x-2t)^2 + \frac{2}{5}(x+3t)^2 + \frac{1}{6}\sin(x+t)$$